Semi-Analytical Surface Tension Model for Free Surface Flows

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ABSTRACT

In this paper, a semi-analytical surface tension model is proposed for smoothed particle hydrodynamics (SPH). Different from previous approaches, cohesive and adhesive forces in our model are unified within a surface energy framework for nonuniform systems. To calculate the adhesive force, we use a semi-analytical solution to convert the volume integral into a surface integral, triangular meshes which represent solid boundaries can be directly introduced into liquid-solid interactions. A gradient descent algorithm is employed to optimize the objective function, which represents the total energy of the fluid. Experiments show that our model can efficiently handle complex solid boundaries with surface-tension-driven phenomena.

Index Terms: Computing methodologies—Computer Graphics—Animation—Physical simulation

1 INTRODUCTION

Various microscopic and macroscopic approaches have been proposed to model surface tension. Since the energy-based surface tension model [6] in SPH framework is more robust at simulating thin features in free surface flows, we extend it to handle both liquid–air and liquid-solid boundaries in a unified way.

To model adhesion between liquid and solid, a straightforward solution is to sample the solid with ghost particles and explicitly model the interaction between liquid and solid as pairwise particle forces [1]. Nevertheless, the particle sampling strategy is neither flexible nor efficient for complex solid boundaries, such as those including corners and shells. Inspired by Chang et al. [3]'s work, we propose a semi-analytical surface tension model to handle liquid-solid interactions. The basic idea is to treat the boundary region as a uniform system.we assume that there is a transitional zone between liquid and solid, where the surface energy is defined as a spatial variation of composition densities. With such an assumption, both cohesion and adhesion can be uniquely derived from a squared gradient energy term. We use triangular meshes to represent solid boundaries, and convert the volume integral over the solid boundary into surface integral on the boundary to calculate the adhesion.

To summarize, our contributions in this paper are:

- An extension of the Helmholtz free energy functional to uniformly model cohesion and adhesion forces at liquid–solid interfaces.
- A semi-analytical solution to calculate the liquid adhesion to the solid boundary, which does not require to sample the solid with ghost particles any more.

This section will give a brief introduction to our semi-analytical surface tension model.

1.1 Semi-analytical Boundary Conditions

Among all boundary handling techniques, ghost particles are most commonly used [7]. Sampling solid boundaries with ghost particles helps maintain the meshfree nature in SPH, however, it is still not an elegant way to deal with large planar regions or deformable boundaries. Instead, semi-analytical methods have been proven to be more efficient at handling large-scale and dynamic boundaries [3]. The basic idea of the semi-analytical solution is to convert a volume integral over the solid into a surface integral over the solid boundary. According to Chang et al. [3], the boundary integral for an arbitrary radial basis function can be formulated as

$$\begin{aligned} f^{\mathscr{B}}(\mathbf{x}) &= \int_{\mathscr{B}} g\left(\mathbf{x},\mathbf{x}'\right) dV \\ &= \int_{\Omega^s} G(h) d\Omega - \int_{\Omega^k} G(r(\theta,\varphi)) d\Omega \end{aligned} , \tag{1}$$

where $\mathscr{B}$ represents the integral domain inside the boundary, Ω^s is the solid boundary, Ω^k is the support domain boundary inside the solid, $r = \|\mathbf{x}-\mathbf{x}'\|$, $g(\mathbf{x},\mathbf{x}')$ represents a radial basis function, i.e., $g(\mathbf{x},\mathbf{x}') = g(r)$, $G(r)$ represents a single variable function satisfying $G'(r) = g(r)r^2$, and $d\Omega = \sin\theta d\theta d\varphi$ represents an expression for the differential in spherical coordinates.

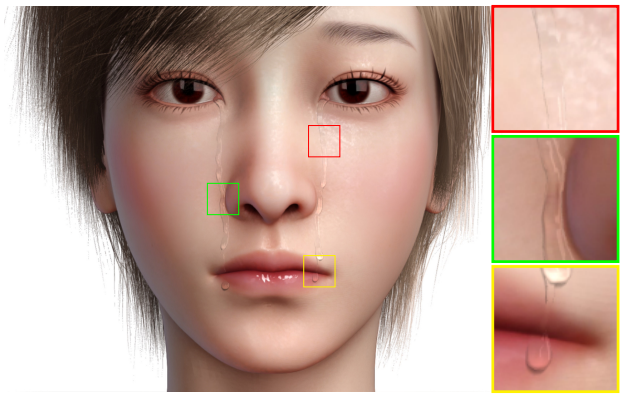

Figure 1: Teardrops. Two particle emitters are used to generate tears, which are then rolling down the cheeks. Our algorithm runs in real-time an FPS range between $20 \sim 60$. Due to the physical interaction, no wet map is required to capture the tear trace.

2 SEMI-ANALYTICAL SURFACE TENSION MODEL

Note that if $G(r)$ satisfies $G(h) = 0$, the integral in Eq. 1 has a simpler form defined as follows

$$f^{\mathscr{B}}(\mathbf{x}) = -\int_{\Omega^k} G(r(\theta,\varphi)) d\Omega \ . \tag{2}$$

The advantage of applying Eq. 2 to take the boundary integral is it removes the necessity to sample solid boundary with ghost particles and a simple surface mesh can be integrated to handle fluid-solid interactions.

2.1 Numerical Discretization

From a mesoscopic view, surface tension forces arise from regions with nonuniform composition [2]. It has been shown that the free

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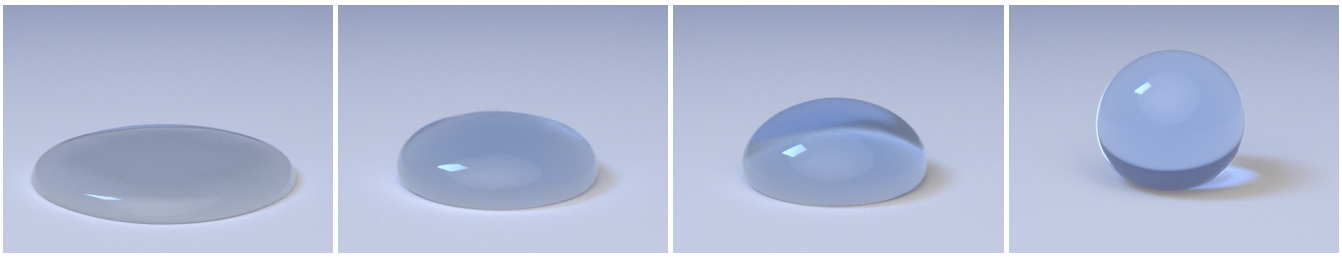

(a) $\kappa^f = 0.01, \kappa^s = 0.075$ (b) $\kappa^f = 0.04, \kappa^s = 0.05$ (c) $\kappa^f = 0.05, \kappa^s = 0.025$ (d) $\kappa^f = 0.06, \kappa^s = 0.01$

Figure 2: Comparison of contact angles with different surface energy coefficients. By adjusting the value of κ^f and κ^s, our method allows simulating different wetting effects. The average simulation time per frame is 16ms.

energy of a volume V for an isotropic system of nonuniform composition is formulated as

$$\mathscr{E} = \int_{\mathscr{F}} \left[f(c) + \frac{\kappa}{2} \|\nabla c\|^2 \right] dV, \tag{3}$$

where $\mathscr{F}$ represents the integral domain, c is the composite density , κ^f is the surface energy coefficient, $f(c)$ is the bulk energy per unit volume of a homogeneous system, and the squared gradient energy term represents the surface energy per unit volume.

Let us consider the area inside the particle's supporting area containing three different materials, i.e., solid, liquid and air. By assuming the air makes no contribution, the surface energy for a three-phase region can be formulated as

$$\mathscr{S} = \int_{\mathscr{F}} \frac{\kappa^f}{2} \|\nabla c^f\|^2 dV + \int_{\mathscr{B}} \frac{\kappa^s}{2} \|\nabla c^b\|^2 dV, \tag{4}$$

where the superscripts f and s represent the surface energy coefficients of the fluid and solid, respectively. To calculate surface forces imposed on a particle i, the first term of Eq. 4 can be calculated by discretizing $\nabla_i c^f$ according to [6], the difficulty lies in how to calculate the surface energy of the solid. Since SPH suffers from the particle deficiency problem, we prefer to apply the corrective smoothed particle method [4] to calculate $\nabla_i c^s$ as follows

$$\nabla_i c^s = \frac{\int_{\mathscr{B}} c_j^s \nabla_{ij} W dV}{\int_{\mathscr{B}} (\mathbf{x}_i - \mathbf{x}_j) \nabla_{ij} W dV}. \tag{5}$$

Note c_i^s has been omitted since its value is zero outside of the solid boundary. By invoking the divergence theorem and the semi-analytical method in Chang et al. [3], the formulation of $\nabla_j c^s$ can be converted into a surface integral defined as

$$\nabla_i c^s = -\frac{\int_{\partial \mathscr{B}} c_j^s W_{ij} d\mathbf{S}}{\int_{\Omega} \tilde{W}_{ij} d\Omega}. \tag{6}$$

where $d\mathbf{S}$ is a shorthand for $\mathbf{n} dS$, $\tilde{W}$ is a weighting function satisfying $\tilde{W}' = r^3 \frac{\partial W}{\partial r}$. Since Equation 6 involves surface integral only, a triangular mesh representing the solid boundary can be incorporated to facilitate the calculation. Therefore, the discrete form of Eq. 6 becomes

$$\nabla_i c^s = -\frac{\sum_j A_j c_j^s \mathbf{n}_j W_{ij} \left(\left\| \mathbf{x}_j - \mathbf{x}_i \right\| \right)}{\sum_j \Omega_j \tilde{W}_{ij} \left(\left\| \mathbf{x}_j - \mathbf{x}_i \right\| \right)}, \tag{7}$$

where A_j represents the area of triangle j, $\mathbf{n}_j$ represents the boundary normal, $\mathbf{x}_j$ represents the closest sampling point of triangle j to $\mathbf{x}_i$, Ω_j represents the solid angle of triangle j with respect to $\mathbf{x}_i$ [3]. Motivated by He et al. [5], the fluid dynamics equipped with surface tension and adhesion can be addressed by solving the following objective function

$$\mathscr{E}_i = \frac{1}{2} \frac{\left\| \mathbf{x}_i - \mathbf{x}_i^* \right\|^2}{d_0^2} + f\left(c_i^f\right) + \frac{\kappa^f}{2} \left\| \nabla_i c^f \right\|^2 + \frac{\kappa^s}{2} \|\nabla_i c^s\|^2, \tag{8}$$

where d_0 denotes the particle sampling distance.Since the total energy is a function of the particle position $\mathbf{x}_i$, we can therefore shift their positions to solve the optimization problem iteratively.

3 Results

We implemented our method with CUDA and run all examples on an NVIDIA graphics card (Geforce RTX2080). To simulate the teardrop in Fig. 1, we place two emitters to produce fluid particles. A total of 67312 triangles and 5070 particles are used to model the simulation. As fluid particles flow down the cheeks, they can interact with the triangular mesh. Note the tear traces resulted from the physical interaction between the tears and the face. Since the performance is not fully optimized, this simulations now runs at an FPS range between $20 \sim 60$. In Fig. 2, a fluid droplet consisting of 2196 particles is simulated to model different wetting conditions.

4 Conclusion

In this study, the energy-based surface tension model was extended to handle liquid-solid interactions, and both cohesion and adhesion forces can be addressed in a unique form.Unlike previous methods, our method does not require us to sample the solid phase and a triangular mesh can be integrated to facilitate calculating surface tension and adhesion.

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